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Penalization and regularization for multivalued pseudo-monotone variational inequalities with Mosco approximation on constraint sets

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Abstract A coupling of penalization and regularization methods for a variational inequality with multi-valued pseudo-monotone operators is given. The regularization permits to include non-coercive operators. The effect of perturbation is also analyzed.

Keywords Variational inequalities · Regularization · Pseudo-monotone

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{H}^*$ be its topological dual which will be identified by $\mathcal{H}$. The associated inner product and the norm will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\Omega \subset \mathcal{H}$ be a nonempty closed and convex set. Let $F : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued map and let $\varphi : \mathcal{H} \to \bar{\mathbb{R}}$.

This work is dedicated to the blessed memory of Professor Alex Rubinov.

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We will focus on the following problem: Find $y \in \Omega$ such that there exists $w \in F(y)$ satisfying

$$\langle w, x - y\rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in \Omega. \tag{1}$$

The above problem is a variational inequality (VI) and $y \in \Omega$ is its solution. We shall denote by $S(F, \varphi, \Omega)$, the set of all solutions of (1).

The field of variational inequalities, initiated by Stampacchia [22], is now a well established branch of pure and applied mathematics with a wide ranging spectrum of its applications (see [23]). Notice that when Ω is a closed convex cone with its apex at the origin and φ is identically zero, the above variational inequality reduces to the well-known complementarity problem, see Ref. [10]. Regularization methods for the complementarity problems have been developed by Isac [9].

In this work we approximate (1) by another VI: Find $x_{\epsilon,\beta} \in \mathcal{H}$ such that there exists $w \in F(x_{\epsilon,\beta})$ satisfying the condition that for every $z \in \mathcal{H}$, the following inequality holds

$$\langle w + \epsilon x_{\epsilon,\beta} + \beta^{-1}\Lambda(x_{\epsilon,\beta}), z - x_{\epsilon,\beta}\rangle \geq \varphi(x_{\epsilon,\beta}) - \varphi(z), \quad \epsilon > 0, \quad \beta > 0, \tag{2}$$

where $\Lambda : \mathcal{H} \to \Omega$ is a penalty operator fulfilling certain requirements that will be discussed shortly.

Although we will use an analogue of (2) that also takes into account perturbations/error in the data for (1), for the time being, it suffices to use (2) to explain the methodology of this work and also to show the differences between our approach and the other approaches available in the literature. In (2), we have coupled the traditional penalization and regularization methods. In fact, the need of the regularization arises as (1), in general, is ill-posed. Besides dealing with some perturbations in the data of (1), the regularization procedure also permits us to relax some conditions on the map F. On the other hand, due to the penalization, (2) does not have the explicit constraint Ω, that is, (2) is defined on the whole of $\mathcal{H}$. This formulation has some advantages in numerical procedures, see Ref. [13].

In recent years the regularization methods for variational inequalities have experienced noteworthy advancements. For example, in Ref. [2] an approximation theory for ill-posed variational inequalities defined by multi-valued monotone operators has been given under very mild conditions, including the notion of Mosco-convergence for the approximation of the constraints sets. Regularization methods for non-monotone operators have been studied in Ref. [12], see also Ref. [3,4,17].

In the short paper, we study a variational inequality with a multi-valued pseudo-monotone operator. The class of pseudo-monotone operators is considerably larger and includes monotone operators. The study of regularization methods for perturbed variational inequalities with single-valued pseudo-monotone operators was initiated by Liscovets [15] (see also Ref. [6]). In an interesting paper Gwinner [8] studied regularization methods for general pseudo-monotone bi-functions. Penalty methods for variational inequalities with single-valued monotone maps are given in Ref. [7] and with multi-valued monotone maps in Ref. [1]. To the best of our knowledge this is the first work that unifies the regularization and/or penalization for variational inequalities with multi-valued pseudo-monotone operators including Mosco's approximation of the underlying constraint sets.

This paper is organized into three sections. In the next section we collect some results to be used in the rest of the paper. In Sect. 3, we deal with the approximation of VI under consideration. The main result ensures that the perturbed VI recovers the original problem in a certain sense.

2 Preliminaries

Let Z be a real reflexive Banach space, let Z^* be the topological dual of Z, let $\langle\cdot,\cdot\rangle_Z$ be the associated pairing and let $\|\cdot\|_Z$ be the norm in Z as well as in Z^*.

We recall the notion of multi-valued pseudo-monotone maps which is due to Browder and Hess [5].

Definition 2.1 A multi-valued map $A : Z \rightrightarrows Z^*$ is said to be pseudo-monotone, if the following three conditions are fulfilled:
(C_1). For each $x \in Z$, the set $A(x)$ is nonempty, bounded, closed and convex.
(C_2). If $x_n^* \in A(x_n)$ is such that x_n converges weakly to x, and the following inequality holds

$$\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle_Z \leq 0,$$

then for each $y \in Z$, there exists $x^*(y) \in A(x)$ such that

$$\liminf_{n\to\infty}\langle x_n^*, x_n - y\rangle_Z \geq \langle x^*(y), x - y\rangle_Z.$$

(C_3). The restriction of A to any finite dimensional subspace M of Z is weakly upper-semi-continuous as a map from M to Z^*.

The following condition (C_4) was introduced by Kenmochi [11] who proved that any operator satisfying (C_1), (C_2), and (C_4), is pseudo-monotone.
(C_4). For each $x_0 \in Z$ and each bounded subset $\mathcal{B}$ of Z, there exists a constant $N(\mathcal{B}, x_0)$ such that

$$\langle x^*, x - x_0\rangle \geq N(\mathcal{B}, x_0), \quad \forall (x, x^*) \in Z \times Z^* \quad \text{with } x^* \in A(x) \text{ and } x \in \mathcal{B}.$$

We conclude this section by recalling an existence result for multi-valued variational inequalities.

Theorem 2.1 [11] *Let $A : Z \rightrightarrows Z^*$ be a multi-valued mapping satisfying (C_1), (C_2), and (C_4), let $C \subset Z$ be nonempty, closed and convex and let $\psi\colon Z \to \mathbb{R}$ be proper convex and lower semi-continuous. Assume that there exists $x_0 \in C$ such that $\psi(x_0) < \infty$ and*

$$\inf_{x^*\in A(x)} \frac{\langle x^*, x - x_0\rangle + \psi(x)}{\|x\|} \to \infty \ \ as\ \ \|x\| \to \infty,\ \ with\ x \in C.$$

Then, for any given $f \in Z^*$, there exists $y \in C$ and $y^* \in A(y)$ such that

$$\langle y^* - f, x - y\rangle \geq \psi(y) - \psi(x), \quad \forall x \in C.$$

3 Main results

We will connect the exact data (F, φ, Ω) to the perturbed data $(F_n, \varphi_n, \Omega_n)$ through the following assumptions:

(A_1). The sequence of nonempty, closed and convex sets Ω_n converges to Ω, in Mosco's sense. That is the following two conditions hold:
  - (a) Ω contains all weak limits of sequences $\{u_k\}$, $u_k \in \Omega_{n_k}$, where Ω_{n_k} is a subsequence of Ω_n.
  - (b) For each $u \in \Omega$ there exists $u_n \in \Omega_n$ such that $u_n \to u$.

(A_2). For each $n \in \mathbb{N}$, the map $F_n : \mathcal{H} \rightrightarrows \mathcal{H}$ satisfies (C_1), (C_2) and (C_4). Moreover, there exists κ: $\mathbb{R}_+ \to \mathbb{R}_+$ which is bounded on bounded sets and is such that for any $x \in \mathcal{H}$, we have

$$\mathcal{H}(F_n(x), F(x)) \leq \alpha_n \kappa(\|x\|), \tag{3}$$

where $\mathcal{H}$ stands for the Hausdorff distance between sets and $\alpha_n > 0$.

(A_3). For each $n \in \mathbb{N}$, the functional φ_n: $\mathcal{H} \to \mathbb{R}$ is proper convex and lower-semicontinuous. There exists ℓ: $\mathbb{R}_+ \to \mathbb{R}_+$ which is bounded on bounded sets and is such that for any $x \in \mathcal{H}$, we have

$$|\varphi_n(x) - \varphi(x)| \leq \delta_n \ell(\|x\|), \quad \delta_n > 0.$$

(A_4) The multi-valued map F satisfies (C_1), (C_2) and (C_4), and the functional φ is convex continuous.

Consider the following Penalized-Regularized Variational Inequality (PRVI): Find $x_n \in \mathcal{H}$ and $w_n \in F_n(x_n)$ such that for every $z \in \mathcal{H}$, the following inequality holds

$$\langle w_n + \epsilon_n x_n + \beta_n^{-1} \Lambda_n(x_n), z - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z), \quad \beta_n > 0, \ \epsilon_n > 0. \tag{4}$$

The operator $\Lambda_n : \mathcal{H} \to \mathcal{H}$ is a penalty operator given by $\Lambda_n(x) = x - P_{\Omega_n}(x)$, where $P_{\Omega_n} : \mathcal{H} \to \Omega_n$ is the nearest point projection map onto Ω_n (see Ref. [20]).

For $n \in \mathbb{N}$, we will denote the set of all solutions of (4) by $S_n(PRVI)$. In the following, the letters k_1, k_2, etc. denote constants.

The following is the main result.

Theorem 3.1 *Assume that* (A_1), (A_2), (A_3), *and* (A_4) *hold. Assume that there are elements* $z_n \in \Omega_n$ *such that* $\|z_n\| < k_1$, $\varphi_n(z_n) < \infty$ *and*

$$\inf_{w_n \in F_n(x)} \langle w_n, x - z_n \rangle \geq k_2 \|x\| + \varphi_n(z_n) - \varphi_n(x), \quad w_n \in F_n(x), \quad x \in \mathcal{H}. \tag{5}$$

Then for every $n \in \mathbb{N}$, *the solution set* $S_n(PRVI)$ *is nonempty. Moreover, if* $\epsilon_n \to 0$, $\alpha_n \to 0$, $\beta_n \to 0$, $\delta_n \to 0$, *and the map* F *is bounded then for every* $x_n \in S_n(PRVI)$, *there exists a subsequence of* $(x_n)_{n\in\mathbb{N}}$ *that converges weakly to some* $x \in S(F, \varphi, \Omega)$.

Proof In view of Theorem 2.1, it suffices to show that for a fixed $n \in \mathbb{N}$, and $w_n \in F_n(x)$ we have

$$\inf_{w_n \in F_n(x)} \frac{\langle w_n + \epsilon_n x + \beta_n^{-1} \Lambda_n(x), x - z_n \rangle + \varphi_n(x) - \varphi_n(z_n)}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

In fact, in view of (5) and the containment $z_n \in \Omega_n$, which implies $\langle \Lambda_n x, x - z_n \rangle \geq 0$, we have

$$\begin{aligned}\langle w_n + \epsilon_n x + \beta_n^{-1} \Lambda_n(x), x - z_n \rangle + \varphi_n(x) - \varphi_n(z_n) &\geq \epsilon_n \|x\|^2 - \epsilon_n \|x\| \|z_n\| + k_2 \|x\|) \\ &\geq \|x\| [\epsilon_n \|x\| - \epsilon_n k_1 + k_2].\end{aligned}$$

This ensures that $S_n(PRVI) \neq \emptyset$. Let us now construct a sequence $(x_n)_{n\in\mathbb{N}}$, by choosing $x_n \in S_n(PRVI)$ arbitrarily. We begin by showing that $(x_n)_{n\in\mathbb{N}}$ is bounded. In view of the definition of x_n, there exists $w_n \in F_n(x_n)$ such that for every $z \in \mathcal{H}$, the following inequality holds

$$\langle w_n + \epsilon_n x_n + \beta_n^{-1} \Lambda_n x_n, z - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z).$$

Recall that there exists $z_n \in \Omega_n$ such that $\varphi_n(z_n) < \infty$ and (5) holds. By setting $z = z_n$ in the above inequality, we obtain

$$\langle w_n + \epsilon_n x_n, z_n - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z_n) + \beta_n^{-1} \langle \Lambda_n(x_n), x_n - z_n \rangle.$$

Notice that for every $y \in \Omega_n$, we have $\langle \Lambda_n(x_n), x_n - y \rangle \geq 0$. (This is a direct consequence of the fact that Λ_n is monotone and vanishes on Ω_n.) Consequently

$$\langle w_n + \epsilon_n x_n, z_n - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z_n).$$

We prove the boundedness of $(x_n)_{n \in \mathbb{N}}$ by contradiction. Assume that $\|x_{n_k}\| \to \infty$ as $k \to \infty$. In view of (5), we have

$$\langle w_n, x_n - z_n \rangle + \varphi_n(x_n) \geq \varphi_n(z_n) + k_2 \|x_n\|.$$

By combining the last two inequalities, we obtain

$$\epsilon_n \langle x_n, x_n - z_n \rangle \leq 0,$$

which implies that

$$\epsilon_n \|x_n\|^2 \leq \epsilon_n \|x_n\| \|z_n\|.$$

Dividing both sides by $\epsilon_n \|x_n\|$ and putting $n = n_k$ yields a contradiction, since $\|z_n\|$ is bounded. This establishes that the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded.

Since the space $\mathcal{H}$ is reflexive, we can extract a subsequence of $(x_n)_{n \in \mathbb{N}}$ converging weakly to some $x^* \in \mathcal{H}$. (We will keep the same notation for the subsequences as well.) We will first show that $x^* \in \Omega$. Recall that the definition of x_n implies that for some $w_n \in F_n(x_n)$, we have

$$\beta_n^{-1} \langle \Lambda_n(x_n), x_n - z \rangle \leq \langle w_n + \epsilon_n x_n, z - x_n \rangle - \varphi_n(x_n) + \varphi_n(z). \tag{6}$$

Notice that,

$$\begin{aligned} \limsup_{n \to \infty} [\varphi_n(z) - \varphi_n(x_n)] &\leq \limsup_{n \to \infty} [\varphi(z) - \varphi(x_n) + \delta_n(\ell(\|z\|) + \ell(\|x_n\|))] \\ &\leq [\varphi(z) - \varphi(x^*)], \end{aligned}$$

where we used the lower-semicontinuity of $\varphi(\cdot)$ and the fact that x_n converges weakly to x^*.

Then, the inequality (6), in view of the fact that $\beta_n \to 0$, shows that

$$\limsup_{n \to \infty} \langle \Lambda_n(x_n), x_n - z \rangle \leq 0.$$

Since the map Λ_n is monotone, we have

$$0 \leq \langle \Lambda_n(x) - \Lambda_n(x_n), x - x_n \rangle$$

We pass to the limit in the above inequality, and use the fact that $P_{\Omega_n} \to P_\Omega$ (see Ref. [21]) provided that $\Omega_n \rightrightarrows \Omega$ in Mosco's sense, to obtain that $\langle \Lambda(x), x - x^* \rangle \geq 0$. By substituting $x = x^* + \lambda z$ where $z \in \mathcal{H}$ is arbitrary, we get $\langle \Lambda(x^* + \lambda z), z \rangle \geq 0$. By letting $\lambda \to 0$, we obtain $\langle \Lambda(x^*), z \rangle \geq 0$ and because this estimate is true for an arbitrary $z \in \mathcal{H}$, we deduce that $\Lambda(x^*) = 0$. Consequently $x^* \in \Omega$.

Finally we proceed to show that $x^* \in S(F, \varphi, \Omega)$, where x^* is a weak limit of a subsequence of $(x_n)_{n \in \mathbb{N}}$. In fact, from the definition of the subsequence $(x_n)_{n \in \mathbb{N}}$, for an arbitrary $z \in \mathcal{H}$, and for some $\bar{w}_n \in F_n(x_n)$, we have

$$\langle \bar{w}_n + \epsilon_n x_n + \beta_n^{-1} \Lambda_n(x_n), z - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z).$$

Since $\Omega_n \rightrightarrows \Omega$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \Omega_n$ which converges strongly to x^*. By substituting $z = u_n$ in the above inequality, we obtain

$$\langle \bar{w}_n + \epsilon_n x_n, u_n - x_n\rangle - \varphi_n(x_n) + \varphi_n(u_n) \geq \beta_n^{-1}\langle \Lambda_n(x_n), x_n - u_n\rangle \geq 0.$$

Due to (A_2) there exists a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in F(x_n)$ satisfying $\|\bar{w}_n - w_n\| \leq \alpha_n\kappa(\|x_n\|) + \frac{1}{n}$. We claim that

$$\limsup_{n\to\infty}\langle w_n, x_n - x^*\rangle \leq 0.$$

By the above estimate, we get

$$\begin{aligned}\langle w_n, x_n - u_n\rangle &\leq \langle w_n - \bar{w}_n, x_n - u_n\rangle + \varphi_n(u_n) - \varphi_n(x_n) + \epsilon_n\langle x_n, u_n - x_n\rangle\\ &\leq \left[\alpha_n k(\|x_n\|) + \frac{1}{n}\right]\|u_n - x_n\| + \varphi_n(u_n) - \varphi_n(x_n) + \epsilon_n\|x_n\|\,\|u_n\| - \epsilon_n\|x_n\|^2.\end{aligned}$$

Using the facts that $\epsilon_n \downarrow 0$, $\alpha_n \downarrow 0$, and the sequences $(x_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are bounded, the above inequality implies that

$$\limsup_{n\to\infty}\langle w_n, x_n - u_n\rangle \leq 0.$$

The above inequality, in view of the fact that $u_n \to x^*$, further implies

$$\begin{aligned}\limsup_{n\to\infty}\langle w_n, x_n - x^*\rangle &\leq \limsup_{n\to\infty}\langle w_n, u_n - x^*\rangle\\ &\leq 0.\end{aligned}$$

Therefore, there exists a subsequence $(w_n)_{n\in\mathbb{N}}$ such that $w_n \rightharpoonup w^*$. Moreover, we have (see Ref. [5]) $w^* \in F(x^*)$ and

$$\lim_{n\to\infty}\langle w_n, x_n\rangle = \langle w^*, x^*\rangle.$$

We will show that

$$\langle w^*, z - x^*\rangle \geq \varphi(x^*) - \varphi(z) \quad \forall z \in \Omega.$$

Let $z \in \Omega$ be arbitrary. We notice the existence of a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in \Omega_n$, converging strongly to z such that for some $\bar{w}_n \in F_n(x_n)$ the following inequality holds:

$$\langle \bar{w}_n + \epsilon_n x_n, z_n - x_n\rangle \geq \varphi_n(x_n) - \varphi_n(z_n).$$

In view of this estimate, we have

$$\begin{aligned}\langle w^*, x^* - z\rangle &= \liminf_{n\to\infty}\langle w_n, x_n - z\rangle\\ &\leq \limsup_{n\to\infty}\langle w_n - \bar{w}_n, x_n - z\rangle + \limsup_{n\to\infty}\langle \bar{w}_n, x_n - z_n\rangle + \limsup_{n\to\infty}\langle \bar{w}_n, z_n - z\rangle\\ &\leq \limsup_{n\to\infty}[\varphi(z_n) - \varphi(x_n)]\\ &\leq \varphi(z) - \varphi(x^*).\end{aligned}$$

Since $z \in \Omega$ is arbitrary, we deduce that $x^* \in S(F, \varphi, \Omega)$. The proof is complete. □

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